Lecture in Osaka University

Correlation Function ,Fourier Analysis and Spectrum

時系列の相関関数、フーリエ解析、スペクトラム *

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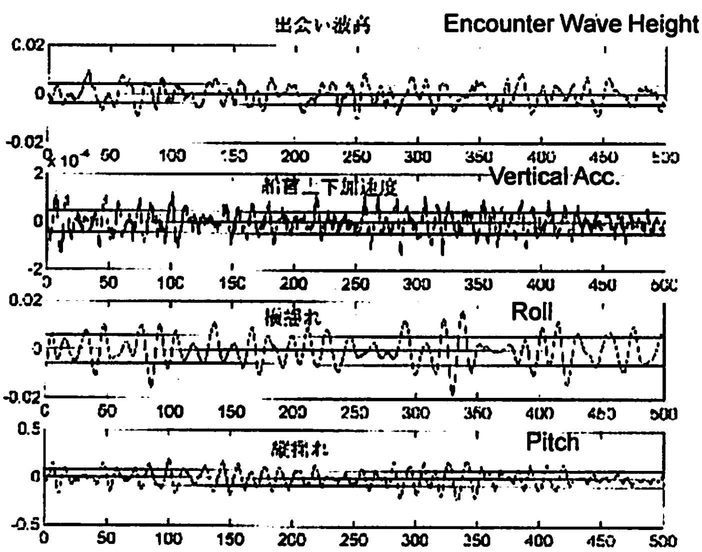


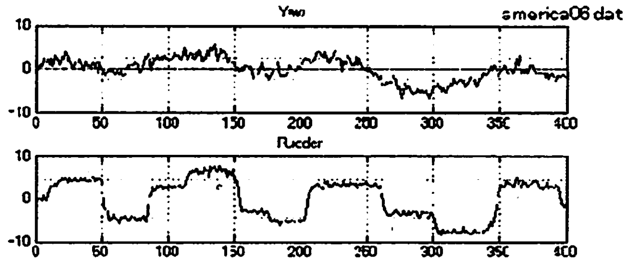


図 2: ある中型コンテナ船の手動操舵の記録

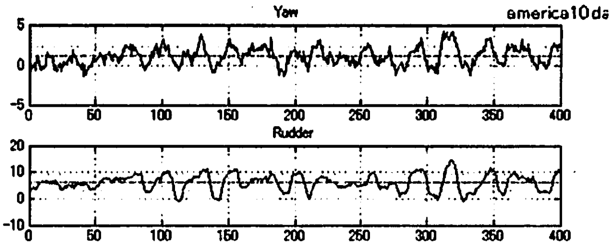


図 3: ある中型コンテナ船のオートパイロットによる航走記録

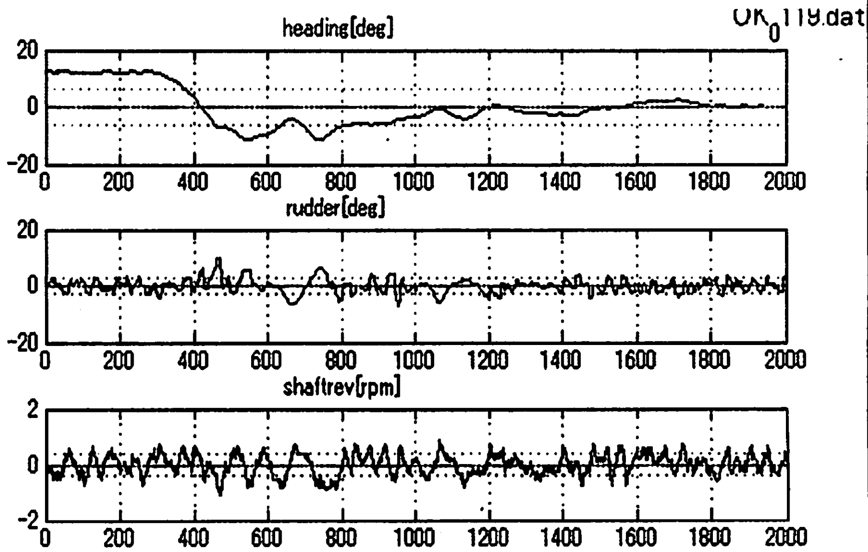


図 4: ある大型コンテナ船の変針時の船首揺れ、舵角、シャフト回転数の記録

Time Series Theory

<u>**The theory**</u>

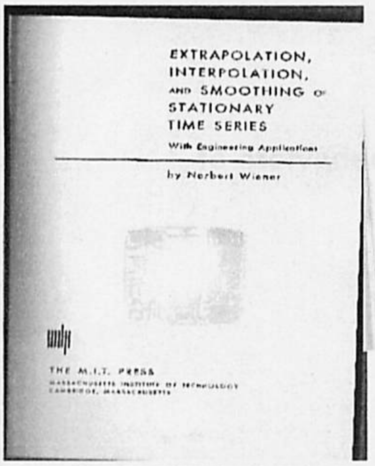
to treat the irregular signals as the stochastic process

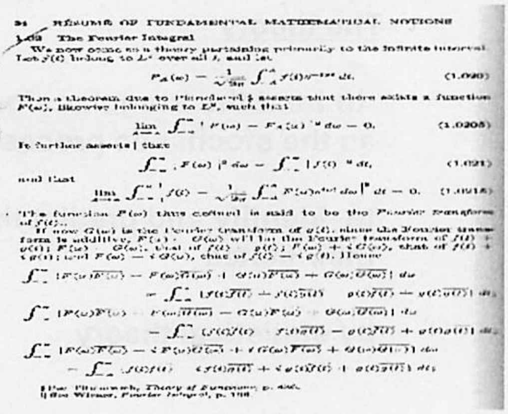
to identify and predict the behaviors of their states

by statistical theory.

Cybernetics

Wiener's Book
The Beginning of Time Series Analysis

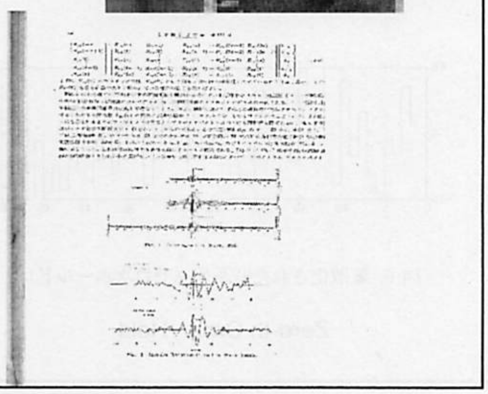




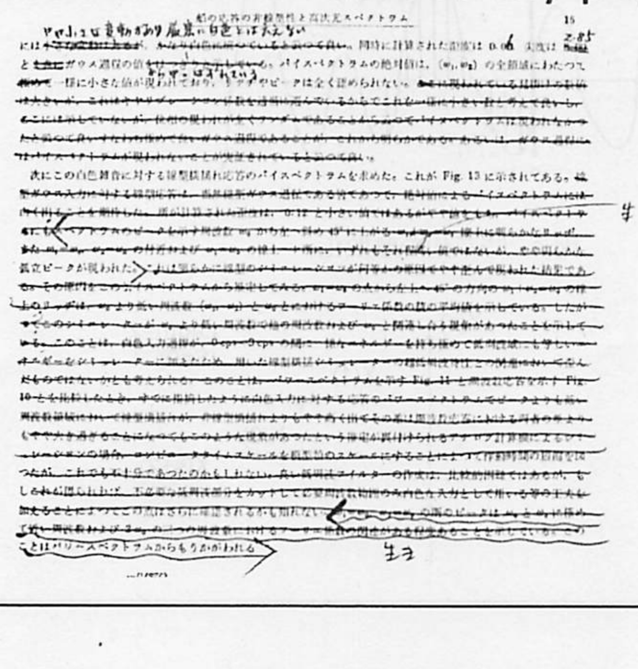
Short History of the study on Time Series Theory

- N.Wiener (1948): Established Basic mathematical theory using the Fourier Analysis
- Blackman and Tukey(1958): Themeasurement of power spectra.' Established the practical method to calculate the power spectra.
- Akaike (1971): Established the time-domain model identification using AIC.

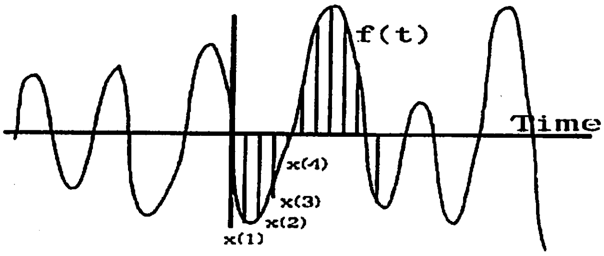
Dr.Y.Yamanouchi



His Proof correction to my paper



Sampling and Histogram of Time Series



Δt
Sampling
Rate

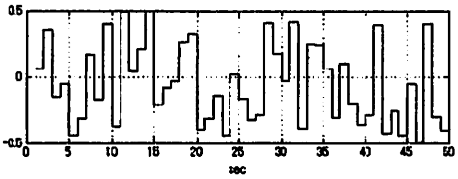
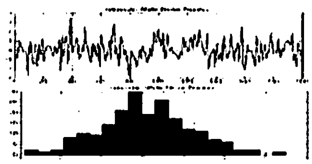


図 6: 離散化された時系列（ゼロ次ホールド）

Zero-th Order Hold



Histogram

Scattering Diagram of Time Series

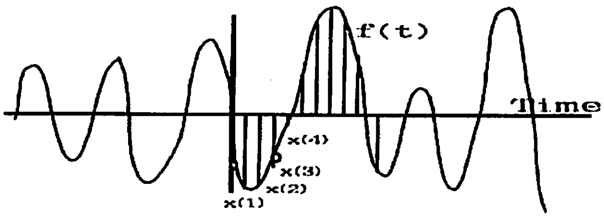
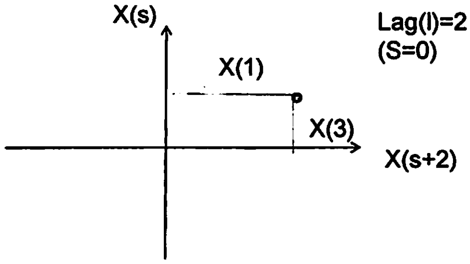


図 5: 時系列の離散化



Lag(l)=2
(S=0)

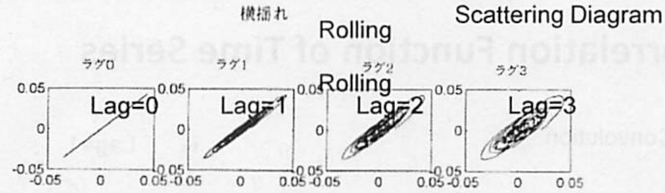


図 8: Rolling の散布図

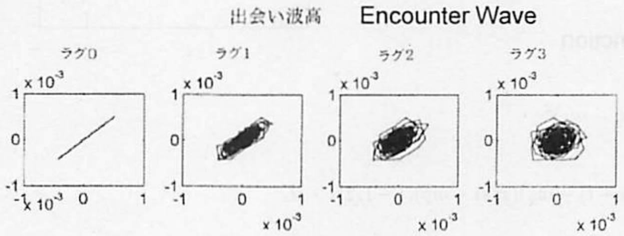


図 9: 出会い波高の散布図

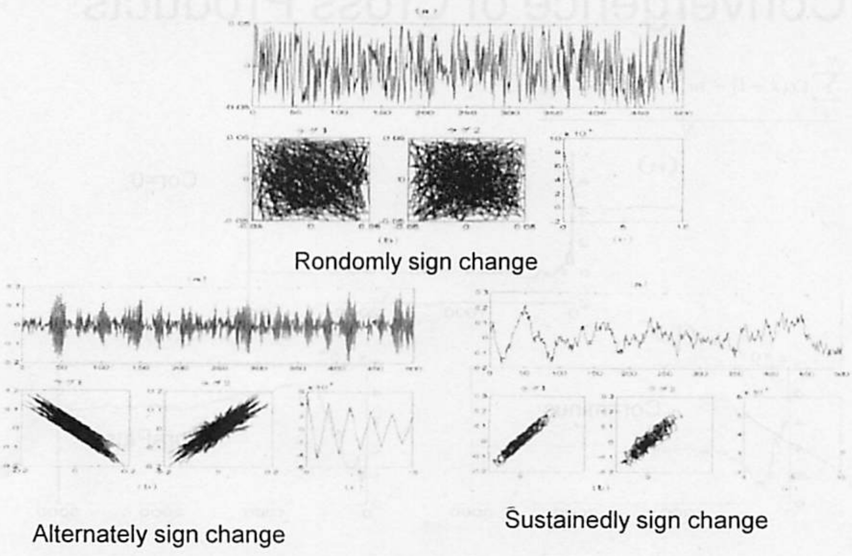
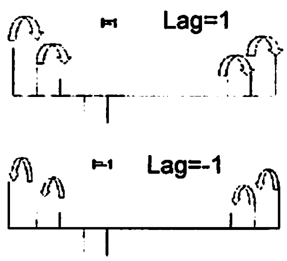


図 10: 種々の散布図 (上：ランダム、下左：交互に異符号　下右：符号持続の時系列)

Correlation Function of Time Series

Mean of Convolution Products

$$\frac{\sum_{t=1}^{N-l}(x(s+l)-m_x)\times(x(s)-m_x)}{N}$$



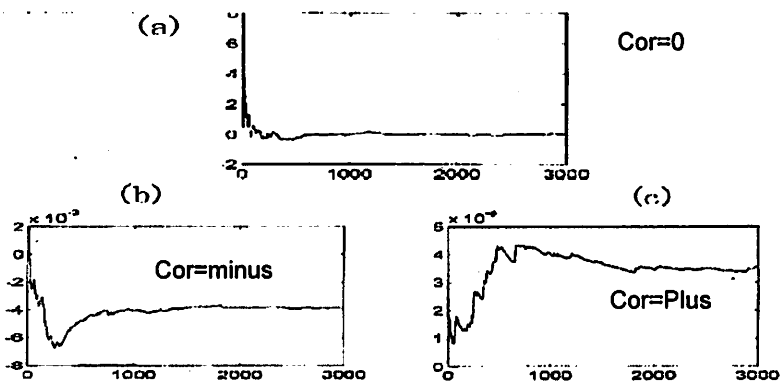
Correlation Function

$$C_{xx}(0)=\frac{1}{N}\sum_{t=1}^{N}(x(s)-m_x)^2$$

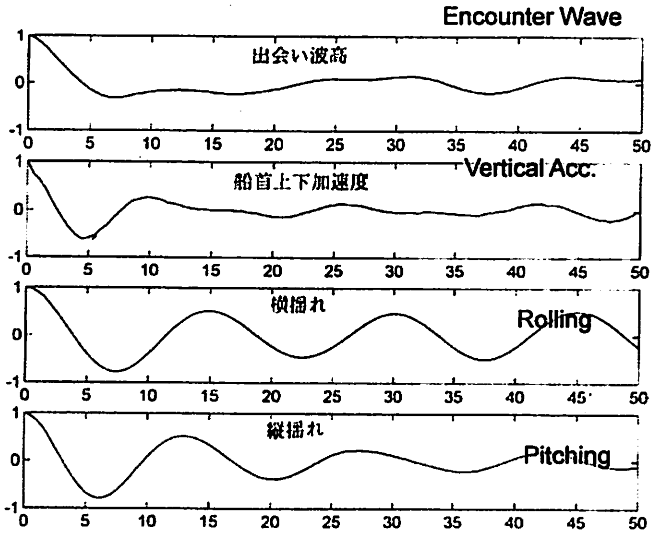
$$C_{xx}(l)=\frac{1}{N}\sum_{t=1}^{N-l}(x(s+l)-m_x)(x(s)-m_x), l-1,2,\cdots,L$$

Convergence of Cross Products

$$\frac{\sum_{t=1}^{N-l}(x(s+l)-m_x)\times(x(s)-m_x)}{N}$$



Examples of Correlation Functions



Sampling

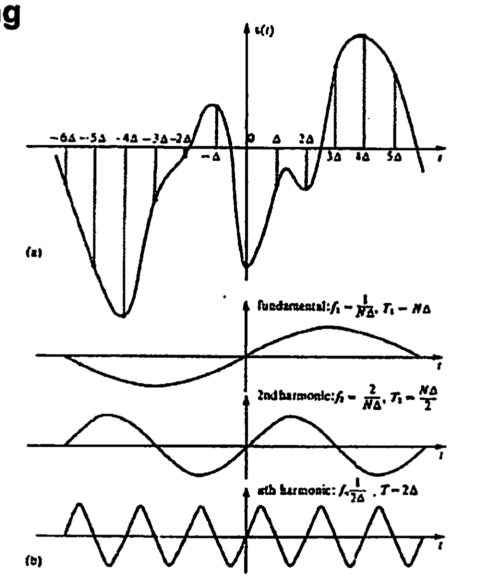


FIG. 2.1: (a) A discrete signal obtained by sampling a continuous signal
(b) The fundamental sine wave and harmonics

Fourier Series Expansion

$[-T/2, T/2]$ $t = 0, \pm T/2n, 2T/2n, \cdots \pm (n-1)T/2n, \pm T/2$

$$x_T(t) = a_0 + 2\sum_{m=1}^{n-1}\left(a_m \cos\frac{2\pi mt}{T} + b_m \sin\frac{2\pi mt}{T}\right) + a_n \cos\frac{2\pi nt}{T}$$

Fundamental frequency $f_1 = 1/T, 2f_1, \cdots$

Discrete Version in Equivalent Time space

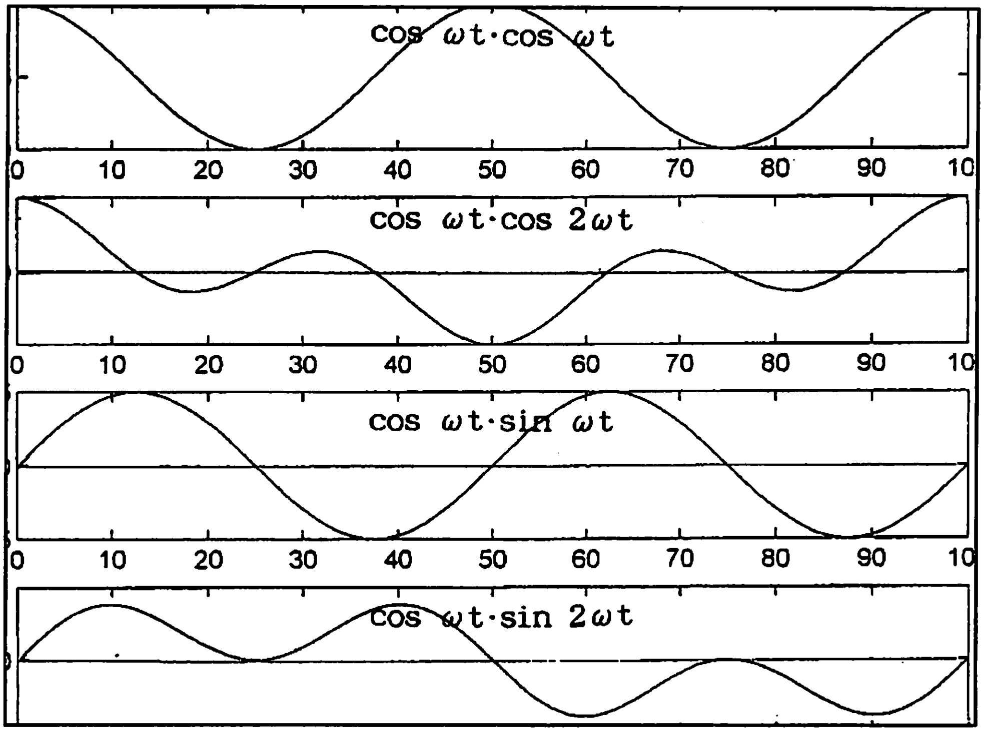
$$\Delta t = T/2n$$

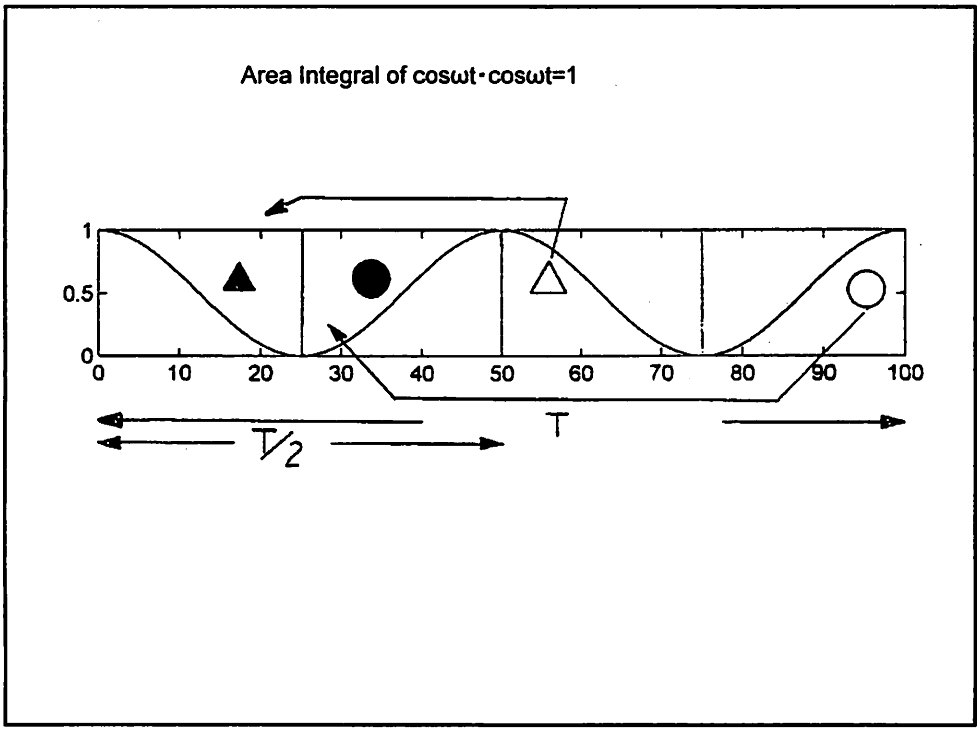
$$x_T(r) = a_0 + 2\sum_{m=1}^{n-1}\left(a_m \cos\frac{2\pi mr}{N} + b_m \sin\frac{2\pi mr}{N}\right) + a_n \cos\frac{2\pi nr}{N}$$

Orthogonality of Trigonometric function

$$\sum_{r=-n}^{n-1} \sin\frac{2\pi kr}{N} \cos\frac{2\pi mr}{N} = 0, \quad k, m = \text{integer}$$

$$\sum_{r=-n}^{n-1} \sin\frac{2\pi kr}{N} \sin\frac{2\pi mr}{N} = \begin{cases} 0 & k \neq m \\ N/2 & k = m \neq 0, n \\ 0 & k = m = 0, n \end{cases}$$


cos ωt·cos ωt
cos ωt·cos 2ωt
cos ωt·sin ωt
cos ωt·sin 2ωt


Area Integral of cosωt·cosωt=1
T/2
T

Fourier Coefficients

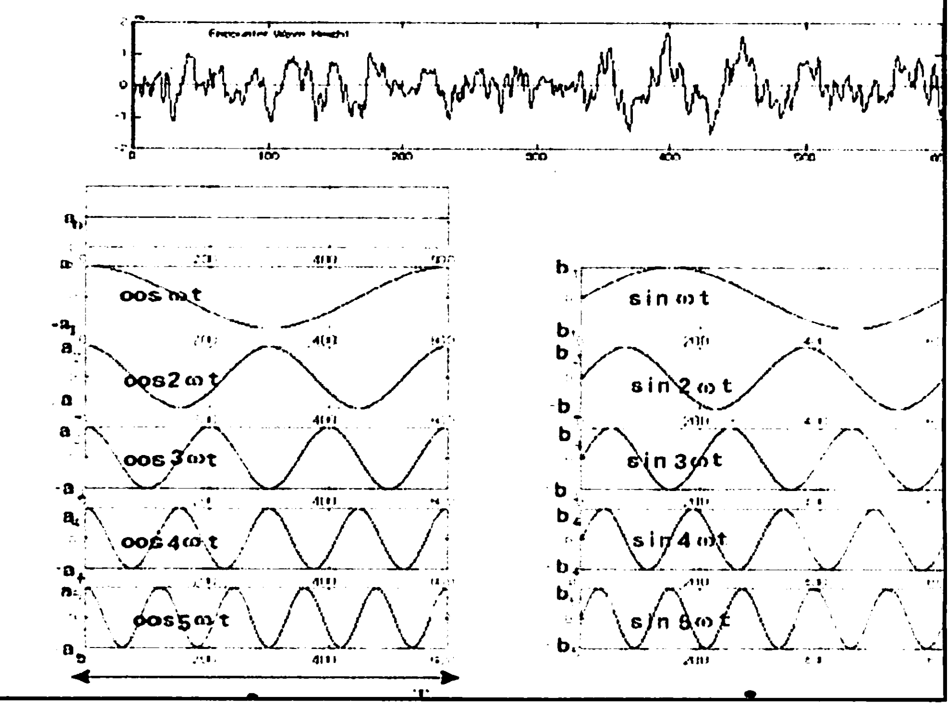
$$\sum_{r=-n}^{n-1} \sin\frac{2\pi kr}{N}\cos\frac{2\pi mr}{N} = 0, \quad k, m = \text{integer}$$

$$\sum_{r=-n}^{n-1} \sin\frac{2\pi kr}{N}\sin\frac{2\pi mr}{N} = \begin{cases} 0 & k \neq m \\ N/2 & k = m \neq 0, n \\ 0 & k = m = 0, n \end{cases}$$

$$x_T(r) = a_0 + 2\sum_{m=1}^{n-1}\left(a_m\cos\frac{2\pi mr}{N} + b_m\sin\frac{2\pi mr}{N}\right) + a_n\cos\frac{2\pi nr}{N}$$

Only Same frequency = not zero!!

$$\begin{aligned} \times\cos\left(\tfrac{2\pi mr}{N}\right) \quad & a_m = \frac{1}{N}\sum_{r=-n}^{n-1} x_T(r)\cos\frac{2\pi mr}{N} && \text{Fourier Cosine Coefficient} \\ \times\sin\left(\tfrac{2\pi mr}{N}\right) \quad & b_m = \frac{1}{N}\sum_{r=-n}^{n-1} x_T(r)\sin\frac{2\pi mr}{N} && \text{Fourier Sine Coefficient} \end{aligned}$$



Fourier Expansion in Continuous Domain

$N \to \infty$ keeping the relation $N\Delta t = T$

$$a_m = \frac{1}{N\Delta t}\sum_{r=-n}^{n-1} x_T(r)\cos\frac{2\pi m r\Delta t}{N\Delta t}\Delta t$$

$$a_m = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)\cos\frac{2\pi m t}{T}dt$$

$$b_m = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)\sin\frac{2\pi m t}{T}dt$$

Fourier Expansion in Continuous Domain

$$x_T(t) = a_0 + 2\sum_{m=1}^{\infty}(a_m\cos\frac{2\pi m t}{T} + b_m\sin\frac{2\pi m t}{T}) \quad (13$$

Parseval's Formulae

Polar Representation

$$R_m = \sqrt{a_m^2 + b_m^2}, \phi_m = \tan^{-1}(-\frac{b_m}{a_m})$$

$$a_m = R_m\cos\phi_m, b_m = -R_m\sin\phi_m.$$

Parseval's Formulae

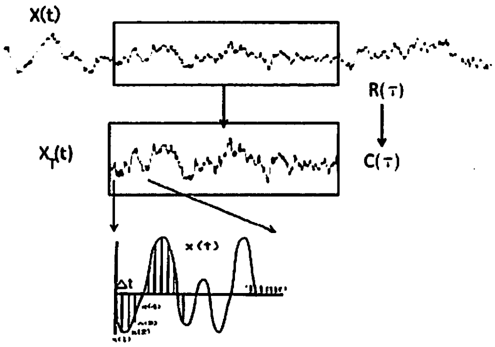
Square of Fourier Amplitude
=Fourier Line Spectrum

$$\frac{1}{N}\sum_{r=-n}^{n-1} x_T(r)^2 = R_0^2 + 2\sum_{m=1}^{n-1} R_m^2 + R_n^2$$

Variance of time series=Energy Power

$$\sigma^2 = \frac{1}{N}\sum_{r=-n}^{n-1}(x_T(r) - R_0)^2 = 2\sum_{m=0}^{n-1} R_m^2 + R_n^2$$

Concept of Sampling



$$x_T(t) = \begin{cases} x(t), -\frac{T}{2} \le t < \frac{T}{2} \\ 0, otherwise \end{cases}$$

Exponential Representation of Fourier Series

From Euler's Relation

$$\sin\eta = \frac{1}{2i}(e^{i\eta} - e^{-i\eta}), \cos\eta = \frac{1}{2}(e^{i\eta} + e^{-i\eta})$$

Eq.13

$$x_T(t) = a_0 + \sum_{m=1}^{\infty}(A_m e^{i\frac{2\pi mt}{T}} + B_m e^{-i\frac{2\pi mt}{T}}) \qquad (20)$$

$$B_{-m} = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-i\frac{2\pi mt}{T}}dt = A_m$$

$$x_T(t) = \sum_{m=-\infty}^{\infty} A_m e^{i\frac{2\pi mt}{T}}$$

$$A_m = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-i\frac{2\pi mt}{T}}dt, -\infty < m < \infty$$

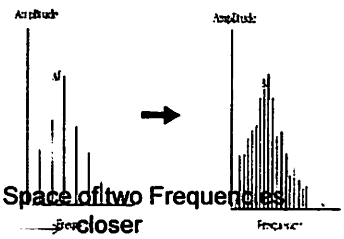
Fourier Integral (First step)

In

$$x_T(t) = \sum_{m=-\infty}^{\infty} A_m e^{i\frac{2\pi m t}{T}}$$

substitute

$$A_m = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-i\frac{2\pi m t}{T}} dt, -\infty < m < \infty$$



Space of two Frequencies closer

$$x_T(t) = \sum_{m=-\infty}^{\infty} \left(\frac{1}{T}\right)\int_{-T/2}^{T/2} x_T(t) e^{-i\frac{2\pi m t}{T}} dt) e^{i\frac{2\pi m t}{T}}$$

$$\delta f_m = f_{m+1} - f_m = \frac{m+1}{T} - \frac{m}{T} = \frac{1}{T}$$

$$x_T(t) = \sum_{m=-\infty}^{\infty} (\int_{-T/2}^{T/2} x_T(t) e^{-i\frac{2\pi m t}{T}} dt) \delta f_m e^{i\frac{2\pi m t}{T}}$$

Fourier Integral (2nd step) $T \to \infty$

$$x_T(t) = \sum_{m=-\infty}^{\infty} (\int_{-T/2}^{T/2} x_T(t) e^{-i\frac{2\pi m t}{T}} dt) \delta f_m e^{i\frac{2\pi m t}{T}}$$

$$A_m = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-i\frac{2\pi m t}{T}} dt, -\infty < m < \infty$$

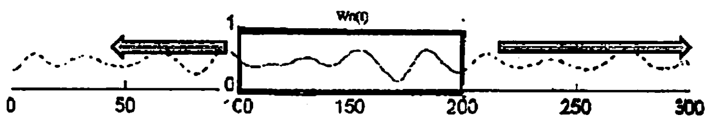
$\frac{m}{T} \to f$ $\quad \delta f_m \to df$

Fourier Transform Pair

$$A(f) = \lim_{T \to \infty} T A_m$$

$$-\infty < t < \infty$$

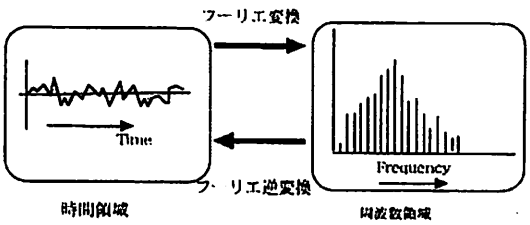
$$x(t) = \int_{-\infty}^{\infty} A(f) e^{i2\pi f t} df \quad \longleftrightarrow \quad A(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$



Time Domain and Frequency Domain

Fourier Transform

$$A(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$



$$x(t) = \int_{-\infty}^{\infty} A(f)e^{i2\pi ft}df$$

Inverse Fourier Transform

Parseval's Formulae

$$\int_{-\infty}^{\infty} x^2(t)dt = \lim_{T\to\infty}\sum_{m=-\infty}^{\infty}|TA_m|^2\frac{1}{T} = \int_{-\infty}^{\infty}|A(f)|^2df$$

Periodogram and Correlation Function

Schuster's Periodogram : Square of Fourier Coefficients

$$I_{xx}(f_m) = N\Delta t|R_m|^2 = N\Delta t|a_{f_m}^2 + b_{f_m}^2|$$

When,

$$d_{fm} = a_{f_m} - ib_{f_m}$$

then

$$I(f_m) = N\Delta t|a_{f_m}^2 + b_{f_m}^2| = N\Delta t d_{f_m} d_{f_m}^*$$

From definition of a and b

$$d_{f_m} = \frac{1}{N}\sum_{r=-n}^{n-1} x_T(r)\cos(2\pi f r\Delta t) - \frac{1}{N}\sum_{r=-n}^{n-1} i x_T(r)\sin(2\pi f_m r\Delta t)$$

$$= \frac{1}{N}\sum_{r=-n}^{n-1}(x(r)e^{-i2\pi f_m r\Delta t})$$

We gain

$$I(f_m) = \frac{\Delta t}{N}\sum_{r=-n}^{n-1} x_T(r)e^{-2\pi f_m r\Delta t}\sum_{q=-n}^{n-1} x_T(q)e^{2\pi f_m q\Delta t}$$

$$= \frac{\Delta t}{N}\sum_{r=-n}^{n-1}\sum_{q=-n}^{n-1} x_T(r)x_T(q)e^{-2\pi f_m(r-q)\Delta t}$$

Algorithm 1 : Ordinary Order

$$I(f_m) = \frac{\Delta t}{N} \sum_{r=-n}^{n-1} \sum_{q=-n}^{n-1} x_T(r) x_T(q) e^{-2\pi f_m (r-q)\Delta t}$$

$x_{-3}x_{-3}$	$x_{-3}x_{-2}$	$x_{-3}x_{-1}$	$x_{-3}x_0$	$x_{-3}x_1$	$x_{-3}x_2$
$x_{-2}x_{-3}$	$x_{-2}x_{-2}$	$x_{-2}x_{-1}$	$x_{-2}x_0$	$x_{-2}x_1$	$x_{-2}x_2$
$x_{-1}x_{-3}$	$x_{-1}x_{-2}$	$x_{-1}x_{-1}$	$x_{-1}x_0$	$x_{-1}x_1$	$x_{-1}x_2$
x_0x_{-3}	x_0x_{-2}	x_0x_{-1}	x_0x_0	x_0x_1	x_0x_2
x_1x_{-3}	x_1x_{-2}	x_1x_{-1}	x_1x_0	x_1x_1	x_1x_2
x_2x_{-3}	x_2x_{-2}	x_2x_{-1}	x_2x_0	x_2x_1	x_2x_2

Algorithm(2) Change of the Order of Summation

$$I(f_m) = \frac{\Delta t}{N} \sum_{r=-n}^{n-1} \sum_{q=-n}^{n-1} x_T(r) x_T(q) e^{-2\pi f_m (r-q)\Delta t}$$

$l = r - q$

$$I(f_m) = \frac{\Delta t}{N} \sum_{l=-N+1}^{N-1} \sum_{r=-n}^{n-1} x_T(r) x_T(r+|l|) e^{-2\pi f_m l \Delta t}$$

1	2	3	4	5	6	$\|l\|$	$\exp -i2\pi \frac{l}{N\Delta t}(\cdot)\Delta t$
$x_{-3}x_2$						5	(5)
$x_{-3}x_1$	$x_{-2}x_2$					4	(4)
$x_{-3}x_0$	$x_{-2}x_1$	$x_{-1}x_2$				3	(3)
$x_{-3}x_{-1}$	$x_{-2}x_0$	$x_{-1}x_1$	x_0x_2			2	(2)
$x_{-3}x_{-2}$	$x_{-2}x_{-1}$	$x_{-1}x_0$	x_0x_1	x_1x_2		1	(1)
$x_{-3}x_{-3}$	$x_{-2}x_{-2}$	$x_{-1}x_{-1}$	x_0x_0	x_1x_1	x_2x_2	0	(0)
$x_{-3}x_{-2}$	$x_{-2}x_{-1}$	$x_{-1}x_0$	x_0x_1	x_2x_1		$\|-1\|$	(−1)
$x_{-3}x_{-1}$	$x_{-2}x_0$	x_1x_{-1}	x_0x_2			$\|-2\|$	(−2)
$x_{-3}x_0$	$x_{-2}x_{-1}$	$x_{-1}x_2$				$\|-3\|$	(−3)
$x_{-3}x_1$	$x_{-2}x_2$					$\|-4\|$	(−4)
$x_{-3}x_2$						$\|-5\|$	(−5)

Wiener Khintine's Theorem

$$I(f_m) = \frac{\Delta t}{N} \sum_{l=-N+1}^{N-1} \sum_{r=-n}^{n-1} x_T(r)x_T(r+|l|)e^{-2\pi f_m l \Delta t}$$

⇓ Correlation Function

$$\frac{1}{N}\sum^{n-1} x_T(r)x_T(r+|l|) \longrightarrow R_{xx}(l)$$

$$I(f_m) = \Delta t \sum_{l=-(N-1)}^{N-1} R_{xx}(l)e^{-i2\pi f_m l \Delta t} \qquad (1.33)$$

The Fourier Transform of Correlation Function=Periodogram=Spectrum(Sxx(f))

$R_{xx}(l) = R_{xx}(-l)$

Simple modification

$$S_{xx}(f) = \Delta t\{R_{xx}(0) + 2\sum_{l=1}^{N-1} R_{xx}(l)\cos 2\pi f l \Delta t\}$$

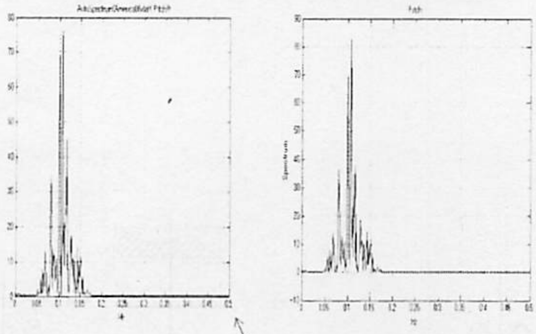
Two Ways to Calculate the Spectrum

1. Data ⟶ Fourier Coeff.: a_{f_m} b_{f_m} ⟶ $I(f_m) = N\Delta t|a_{f_m}^2 + b_{f_m}^2|$ ↓ Spectrum

2. Data ⟶ Correlation Function ⟶ Fourier Transform ↑ Spectrum

Raw spectrum

Periodogram Method　　　　Correlation Method

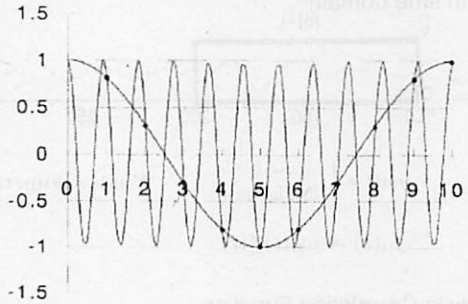


$\frac{1}{2\Delta t}$

Nyquist Frequency

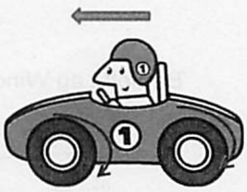
Problem : Big Change

The Reason for Big Change in Raw Spectrum　1 Aliasing



Large
Sampling rate

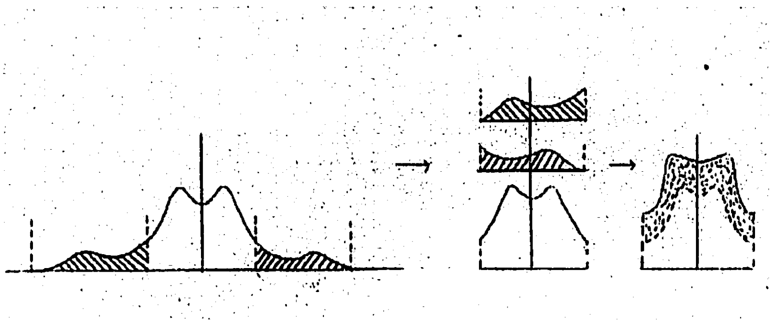
$\frac{1}{2\Delta t}$



$$\sin\frac{2\pi k}{N}n = \sin\frac{2\pi(k+N)}{N}n$$

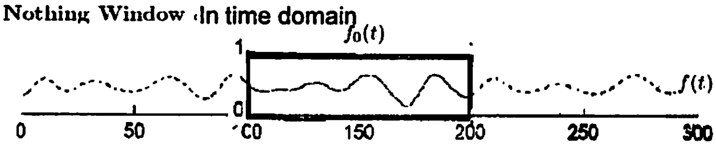
$$\cos\frac{2\pi k}{N}n = \cos\frac{2\pi(k+N)}{N}n, n = 0, 1, 2, \cdots$$

Aliasing（偽名 False Name）



The Reason for Big Change in Raw Spectrum(2)　Window Effect

Do Nothing Window -In time domain

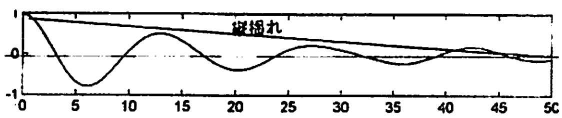


$$w_0(t) = \begin{cases} 1; a < t < b \\ 0; otherwise \end{cases}$$ Window Function

$$f_0(t) = w_0(t) f(t)$$

Bartlett Lag Window in Correlation Function

$$w_0(u) = \begin{cases} (1 - \frac{|u|}{T}), 0 \le |u| \le T \\ 0, |u| > T \end{cases}$$



Larger lag　smaller points

Do-nothing Window Effect

From Convolution Theorem in Fourier transform

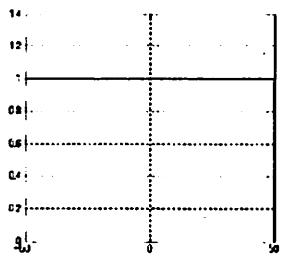
Lag Window

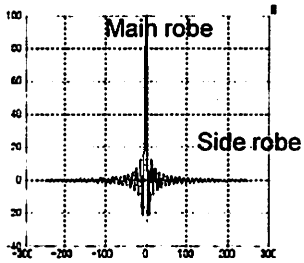
Spectral Window

$$F_0(f) = \int_{-\infty}^{\infty} w_0(t) f(t) e^{-i2\pi f t} dt = \int_{-\infty}^{\infty} W_0(f-f') F(f') e^{-i2\pi f' t} df$$

Case : Periodogram

Boxcar Shape

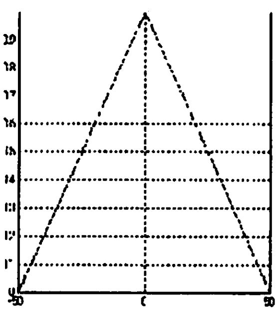


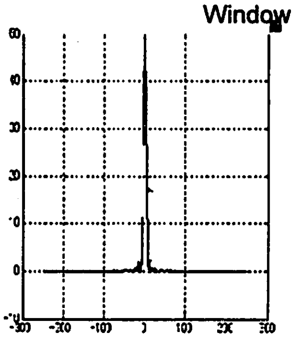


Case : Correlation Method

$$E[S_{xx}(f)] = \int_{-\frac{T}{2}}^{\frac{T}{2}-|u|} \left(1-\frac{|u|}{T}\right) R_{xx}(u) e^{-i2\pi f u} du$$

Bartlett Triangle Window





Lag Window

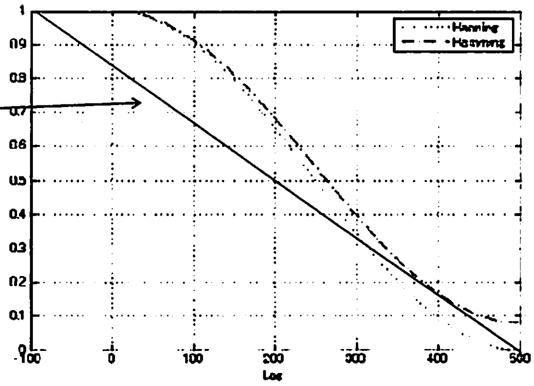
$$E[S_{xx}(f)] = \int_{-\frac{T}{2}}^{\frac{T}{2}-|u|} \left(1-\frac{|u|}{T}\right) R_{xx}(u) e^{-i2\pi f u} du$$

Positively changing

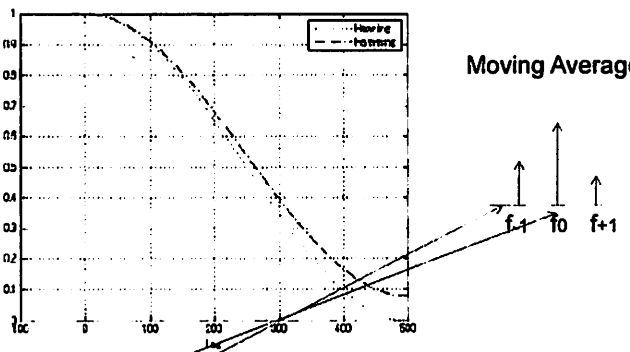
$$w_{Han}(l) = \begin{cases} \frac{1}{2}\left(1+\cos\frac{\pi l}{L}\right) & l \le L \\ 0 & l > L \end{cases}$$ Hanning Window

$$w_{Ham}(u) = 0.54 + 0.46\cos\frac{\pi l}{L}$$ Hamming Window

Bartlett Window



Moving Average Window in Spectrum



Moving Average

	Hanning	*Hamming*	W_1	W_2	W_3
a_0	0.50	0.54	0.5132	0.6398	0.7029
$a_1 = a_{-1}$	0.25	0.23	0.2434	0.2401	0.2228
$a_2 = a_{-2}$	0.0	0.0	0.0	−0.0600	−0.0891
$a_3 = a_{-3}$	0.0	0.0	0.0	0.0	0.0149

Smoothed Spectrum of Rolling